AN ADAPTIVE RECURSIVE WEIGHTED FILTER BASED ON THE MYRIAD OPERATOR

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Abstract. *In this paper, we introduce a new family of nonlinear filters based on the Myriad operator. The proposed approach introduces recursivity into the filtering operation where previous filter outputs are fedbacked into the filtering operation to conform an observation window that has input samples and previous filtered values leading thus to a recursive Weighted Myriad filter (RWMy). The proposed filtering structure is equipped with a tunning parameter that allow to set the degree on impulsiveness rejection capability. Thus, as that parameter goes to infinity the RWMy reduces to the well-known Infinity Impulse response (IIR) filter which has been proven to outperform the finite impulse response (FIR) filter. Furthermore, an adaptive optimization algorithm based on equation error formulation is also proposed for the design of RWMy where an absolute error cost function is minimized at each iteration. Extensive computer simulations show that the proposed approach outperforms the no recursive version much like the IIR outperforms the FIR filters.*

Keywords: Recursive weighted myriad, Maximum likelihood, Equation error formulation, Adaptive filter theory, Nonlinear filter.

1 INTRODUCTION

It is well–known that signal filtering in real–environment scenario where signal's contaminations exhibit a higher impulsiveness than that described by the Gaussian noise model demands the use of a robust filtering operation suitable for noise modeled by heavy-tailed distribution. Weighted Myriad Filter (WMyF), for instance, emerges as a robust nonlinear filtering approach that is derived as a generalization of the sample myriad [1]. More precisely, given N observation samples $\{x_i\}_{i=1}^N$ and weights $\{w_i\}_{i=1}^N$ the weighted myriad is the maximum likelihood estimator of location parameter for the class of Cauchy distribution — a particular case of the family of α–stable distribution. WMyF takes advantage of a linearity parameter, that, in a tunning fashion way, adds to the filtering operation the capability to adapt to several degrees of impulsive noise rejection [1, 2]. Following this line of thought, in this paper we propose a new recursive weighted myriad filter under the framework of a noncausal system where given N observation input samples $\{x_i\}_{i=1}^N$ weighted by $\{g_i\}_{i=1}^N$ and M previous filtered outputs $\{y_j\}_{j=1}^M$ with weights $\{h_j\}_{j=1}^M$ the recursive WMyF outputs the maximum likelihood estimation for the location parameter fo the Cauchy distribution of the joint data samples $\{x_i|_{i=1}^N,\ y_j|_{j=1}^M\}$ weighted, respectively, by $\{g_i|_{i=1}^N,\ h_j|_{j=1}^M\}$. Much like recursive linear filters (IIR) offer several advantages over the nonrecursive counterpart (FIR filter), the proposed recursive WMyF also exhibit superior performance than the one found with nonrecursive WMyF [3]. Furthermore, equipped with tunning parameters the proposed filtering approach can be suitably adapted to different degree of impulsiveness of the background noise. However, the fact that the output of a Myriad based estimator is restricted to the dynamic range of the input samples affects the performance of the new filtering framework leading to an undesirable attenuation of the filter outputs [1]. To overcome this apparent limitation the previous filtered output are scaled as they are introduced in the Myriad operation avoiding thus the attenuation on the filter's output. Furthermore, we propose an adaptive optimization algorithm for the design of this new family of nonlinear filter, under the framework of equation error formulation where the previous filter outputs are replaced by the previous desired samples leading to a two–input single output system during the learning stage. The performance of the proposed approach is tested in several signal processing tasks that involve the design of frequency selective filters. It is shown that the proposed approach yields much better performance than the linear counterpart (IIR) and it has similar performance than the WMyF at a much lower computational cost.

2 PRELIMINARY BACKGROUND

Impulsive noise in real–environment scenario can be statistical characterized by heavier–than–Gaussian tail distribution. One of the statistical model most widely used is the α-stable distribution family supported, perhaps, by the Generalized Central Limit Theorem. A member of this distribution family that is of particular interest is the Cauchy distribution since it is the only heavy tail distribution with close form expression for the probabilistic density function (pdf) for which an estimation of parameter location has been derived [4]. To be more precise, let's model the observation sample as $x_i = \theta + \eta_i$, $i = 1, 2, \ldots, N$, where the common parameter θ is a unknown location parameter to be estimated and η_i represents independent and identically distributed (i.i.d.) impulsive noise added to the signal of interest. By involving the maximum likelihood estimation (MLE) approach, the signal of interest can be estimated as $\hat{\theta} = \arg\max \prod_{i=1}^N f_\eta(x_i - \theta)$, where $f_\eta(x)$ is the distribution function followed by the additive noise. That is, for the Cauchy distribution $f(x) = (\gamma/\pi)[1/(\gamma^2 + x^2)]$, where γ is the dispersion parameter of the distribution.

It has been shown in [1] that the MLE of location parameter is given by the myriad operator. That is

$$\hat{\theta}_k = \arg\min_\theta \sum_{i=1}^N \log\left[k^2 + (x_i - \theta)^2\right] = \text{myriad}\ (x_i|_{i=1}^N; k). \tag{1}$$

where k is a linearity parameter that gives to the myriad operator the capability of tuning its impulsive rejection property, ranging from the mean operator of the observation sample for $k \to \infty$ to the mode operator of $\{x_i|_{i=1}^N\}$ for $k \to 0$.

A more general expression for the myriad operator that makes it suitable for a wide variety of filtering processes is the weighted myriad (WMy) framework [4]. That is defined as

$$\hat{\theta}_k = \arg\min_{\theta} \sum_{i=1}^{N} \log\left[k^2 + |w_i|(sgn(w_i)x_i - \theta)^2\right] = \text{myriad}\left(|w_i| \circ sgn(w_i)x_i|_{i=1}^{N}; k\right) \qquad (2)$$

where $|w_i| \circ sgn(w_i)x_i$ denotes the weighting process of the WMy framework [5], that allows to capture the statistical relationships among different samples in an observation signal window where the value given to w_i is related to some degree of reliability of the i_{th} sample [1].

Interestingly, if the contamination follows a Gaussian distribution[1] the WMy behaves as a linear FIR filter as the tunable parameter k becomes infinity. $\hat{\theta}_{k\to\infty} = \sum_{i=1}^{N} w_i' \cdot x_i$, where $w_i' = w_i / \sum_{i=1}^{N} |w_i|$.

Note that the filter output depends on the input observation samples x_i and the filter weights that defines the kind of filtering operation. A more general linear filtering approach that uses not only input samples but also previous computed outputs to define the current filter output is the class of filter with infinite impulse response (IIR), leading to recursive linear filter that has better frequency response than the nonrecursive counterpart.

Much like FIR filters can be extended to a recursive version — the IIR filters. It is natural to think that the weighted myriad filter (WMyF) which is supported by similar principles can be extended to a richer filtering structure, namely the Recursive Weighted Myriad Filter (RWMyF).

3 RECURSIVE WEIGHTED MYRIAD FILTER

The introduction of previously computed outputs of the filtering in the Myriad framework leads to the definition of a more robust filter with a high degree of accuracy and a reduced number of parameters. Formally,

Definition 1.1: *Given a set of observation samples* $\{x_i = x(n+i)|_{i=N_1}^{N_2}\}$ *and a set of previous computed outputs* $\{y_j = y(n-j)|_{j=1}^{M}\}$, *with weights* $\{g_i|_{i=N_1}^{N_2}\}$ *and* $\{h_j|_{j=1}^{M}\}$, *and linearity parameters* k_1 *y* k_2. *The Recursive Weighted Myriad Filter is defined as*

$$\begin{aligned} y_{k_1,k_2}(n) &= \arg\min_{\theta} \sum_{i=-N_1}^{N_2} \log\left[k_1^2 + |g_i|(sgn(g_i)x_i - \theta)^2\right] + \sum_{j=1}^{M} \log\left[k_2^2 + |h_j|(sgn(h_j)y_j - \theta)^2\right] \\ &= \text{myriad}\left(|g_i| \circ sgn(g_i)x_i|_{i=N_1}^{N_2}; |h_j| \circ sgn(h_j)y_j|_{j=1}^{M}; k_1; k_2\right). \end{aligned} \qquad (3)$$

Figure 1(a) illustrates graphically the proposed filtering structure. Note that previous filter outputs are fed back into the myriad operator to define a new output.

One of the strongest limitations of equation (3) and its output is the passiveness of the system and the attenuation associated to this framework. To overcome this apparent limitation, it is proposed the Scaled RWMy where the output is defined as $y(n)^{(S)} = \left(\sum_{i=-N_1}^{N_2} |g_i| + \sum_{j=1}^{M} |h_j|\right) \hat{\theta}_{k_1,k_2}$, and allows to break the unconstrained representation given by (3) and its normalization.

[1] It is well-known that under MLE principle the optimum filtering operator for additive Gaussian noise is a linear FIR kind of filter.

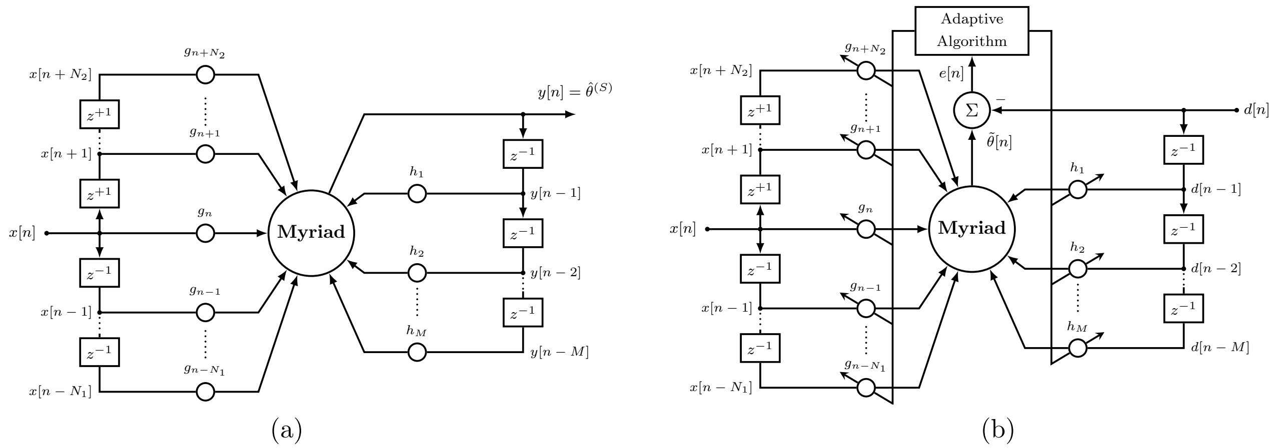


Figure 1: (a) Recursive weighted myriad filter. (b) Adaptive training structure.

4 ADAPTIVE RECURSIVE WMy FILTERING ALGORITHM

Most applications that demand filtering operations need to design optimal values for their weights. To this end, we adopt an adaptive approach where the RWMy filter's weights are found such that a cost function is minimized iteratively following the steepest descent approach. Taking the mean absolute error (MAE) between the filter's output $y(n)$ and a desired signal $d(n)$ as a cost function, i.e.

$$J(g, h; k_1, k_2) = E\{|y(n)^{(S)}[n] - d[n]|\} \tag{4}$$

where $E\{\cdot\}$ represents the statistical expectation. However, taking the derivative of (4) with respect to g_i and h_j with $i = -N_1, -N_1+1, \ldots, N_2$ and $j = 1, 2, \ldots, M$ becomes mathematically intractable since the recursive operation leads to successive application of chain rules. To overcome this drawback we follow the equation error formulation where the previous filter outputs are replaced by previous desired signal ($y(n-i) = d(n-i)$) breaking in that way the recursion and leading to a two-input single output $[\tilde{y}(n)^{(S)}]$ system without recursion as shown in Fig. 1(b).

Following the steepest-descendent approach, it can be shown taht the filter weights can be update as

$$g_i[n+1] = g_i[n] - \mu sgn(e[n]) \left[sgn(g_i[n])\tilde{\theta}_{k_1,k_2}[n] + \left(\sum_{i=-N_1}^{N_2} |g_i[n]| + \sum_{j=1}^{M} |h_j[n]| \right) \frac{\partial \tilde{y}_{k_1,k_2}}{\partial g_i}[n] \right]$$

$$h_j[n+1] = h_j[n] - \mu sgn(e[n]) \left[sgn(h_j[n])\tilde{\theta}_{k_1,k_2}[n] + \left(\sum_{i=-N_1}^{N_2} |g_i[n]| + \sum_{j=1}^{M} |h_j[n]| \right) \frac{\partial \tilde{y}_{k_1,k_2}}{\partial h_j}[n] \right] \tag{5}$$

where $sgn(\cdot)$ represent the sign function. $g_i[n]$ and $h_j[n]$ are the ith and jth filter's weights iteration at the n_{th} and $\mu > 0$ is the step-size parameter.

It is shown in [6] that the partial derivative of $\tilde{y}(n)$ with respect to g_i, h_j are respectively,

$$\frac{\partial \tilde{\theta}_{k_1,k_2}}{\partial g_i} = - \frac{\dfrac{k_1^2 sgn(g_i)(\tilde{\theta} - sgn(g_i)x_i)}{[k_1^2 + |g_i|(sgn(g_i)x_i - \tilde{\theta})^2]^2}}{\displaystyle\sum_{i=-N_1}^{N_2} |g_i| \frac{k_1^2 - |g_i|(sgn(g_i)x_i - \tilde{\theta})^2}{[k_1^2 + |g_i|(sgn(g_i)x_i - \tilde{\theta})^2]^2} + \sum_{j=1}^{M} |h_j| \frac{k_2^2 - |h_j|(sgn(h_j)d_j - \tilde{\theta})^2}{[k_2^2 + |h_j|(sgn(h_j)d_j - \tilde{\theta})^2]^2}} \tag{6}$$

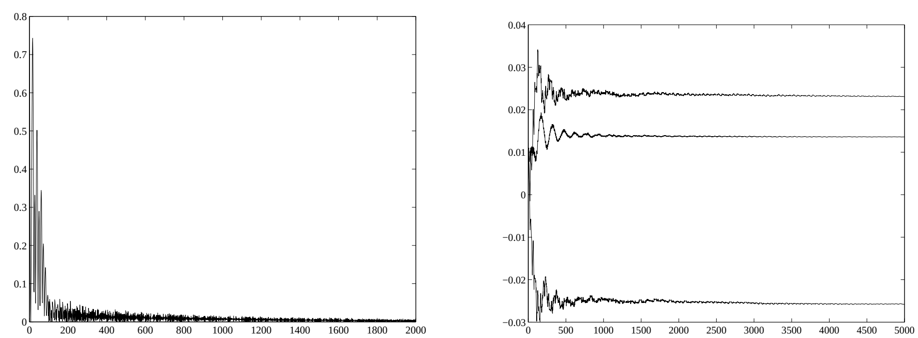
$$\frac{\partial \tilde{\theta}_{k_1,k_2}}{\partial h_j} = - \frac{\dfrac{k_2^2 sgn(h_j)(\tilde{\theta} - sgn(h_j)d_j)}{[k_2^2 + |h_j|(sgn(h_j)d_j - \tilde{\theta})^2]^2}}{\displaystyle\sum_{i=-N_1}^{N_2} |g_i| \frac{k_1^2 - |g_i|(sgn(g_i)x_i - \tilde{\theta})^2}{[k_1^2 + |g_i|(sgn(g_i)x_i - \tilde{\theta})^2]^2} + \sum_{j=1}^{M} |h_j| \frac{k_2^2 - |h_j|(sgn(h_j)d_j - \tilde{\theta})^2}{[k_2^2 + |h_j|(sgn(h_j)d_j - \tilde{\theta})^2]^2}} \tag{7}$$

5 SIMULATION RESULTS

This section develops computer simulation examples involving the design of a bandpass filter.

Considering a three tones signal $s[n] = \sum_{k=1}^{3} a_k \sin(2\pi f_k n)$ with amplitudes $a_1 = 3$, $a_2 = 1$ and $a_3 = 0,6$, Nyquist sample frequency of 1kHz, normalized frequencies $f_1 = 0,02$, $f_2 = 0,09$ and $f_3 = 0,2$. The desired signal $d[n]$ is represented by the central frequency signal $d[n] = a_2 \sin(2\pi f_2 n)$. The additive noise that contaminates the input signal is modeled by α-stable noise with $\alpha = 0,8$ and $\gamma = 0,1$. In this simulation we want to design 82 taps RWMy filter with $N_1 = 25$, $N_2 = 27$ and $M = 30$ and compare its performance to those yielded by FIR, IIR and nonrecursive WMy filter designed for the same filtering task with 82 parameters. The FIR and IIR filter taps are designed using the *fir1* and *yulewalk* MATLAB functions, respectively, with cutoff normalized frequencies of $f_{c_1} = 0.05$ and $f_{c_2} = 0.13$. The nonrecursive WMy is designed through the algorithm proposed in [4] with $k_1 = 0,57$. Furthermore, the scaled RWMy filter is designed using the proposed adaptive approach, expressions (5), (6) and (7) with $\mu(n) = 0,001$ for $n < 100$, a step-size function as $\mu(n) = 0,001\exp(-n/1000)$ for $n > 100$ and $k_1 = k_2 = 0,57$. The results of filtering the signal $x[n] = s[n] + \eta[n]$ with the FIR, IIR, WMy and RWMy filters are shown in Fig. 3.

In all representations of Fig. 3 involve a clean first half, or without noise, and the last part with α-stable noise. Figure 3(a) shows the input signal to the filtering frameworks and the desired signal in Fig. 3(b). From Fig. 3(c) and 3(d), it can be seen it the good performance of linear representations to a clean signal, but the poor performance to a corrupted heavy-tailed noise of input. Finally, the weighted myriad filter shown in figure 3(e) allows a good rejection of input noise but with a strong low pass characteristic. However, it is the scaled approach, figure 3(f), which offers the best estimation and the most similar output to a desired signal.



(a) (b)

Figure 2: Band Pass Filter. (a) Error. (b) Weights: g_{15}, h_5 and g_{25}.

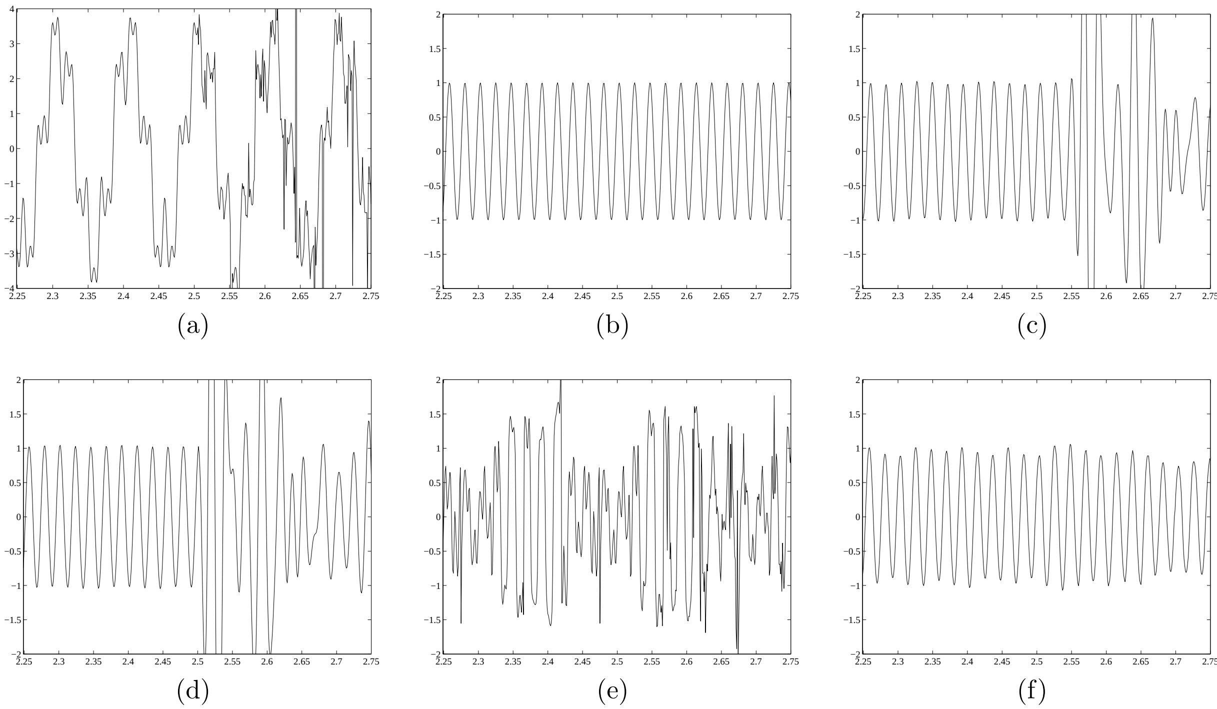


Figure 3: Design of a Band Pass Filter. (a) Input corrupted signal. (b) Desired signal (c) FIR filter output (d) IIR filter output (e) Nonrecursive WMy filter output (f) RWMy filter output.

Thus, the performance of the scaled framework is leaded by the convergency of the parameters presented in the figure 2. Figure 2(a) shows the minimization of the equation error formulation presented in equation (4). Likewise, figure 2(b) represent the adaptability of the parameters g_{15}, h_5 and g_{25} in decremented order, respectively.

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